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Passive trapped modes in the water-wave problem for a floating structure

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Trapped modes in the linearized water-wave problem are free oscillations of an unbounded fluid with a free surface that have finite energy. It is known that such modes may be supported by particular fixed structures, and also by certain freely floating structures in which case there is, in general, a coupled motion of the fluid and structure; these two types of mode are referred to respectively as sloshing and motion trapped modes, and the corresponding structures are known as sloshing and motion trapping structures. Here a trapped mode is described that shares characteristics with both sloshing and motion modes. These 'passive trapped modes' are such that the net force on the structure exerted by the fluid oscillation is zero and so, in the absence of any forcing, the structure does not move even when it is allowed to float freely. In the paper, methods are given for the construction of passive trapping structures, a mechanism for exciting the modes is outlined using frequency-domain analysis, and the existence of the passive trapped modes is confirmed by numerical time-domain simulations of the excitation process.

1. Introduction

Investigations into the uniqueness of solutions to linearized inviscid water-wave problems have resulted in the discovery of trapped-mode oscillations. A trapped mode is a free oscillation of an unbounded fluid with a free surface that has finite energy, does not radiate waves to infinity and persists for all time in the absence of viscosity. Trapped modes occur at isolated frequencies and can be supported only by special trapping structures. Two types of trapped mode have previously been identified: sloshing and motion trapped modes. A sloshing trapped mode is a free oscillation of the fluid around a particular 'fixed' structure referred to as a sloshing trapping structure. On the other hand, a motion trapped mode is a free oscillation of the fluid around a floating structure that is free to move and, in general, involves a coupled motion of the fluid and structure – such a structure is termed a motion trapping structure. McIver (1996) proved the existence of sloshing trapped modes while investigating uniqueness in scattering and radiation problems, while McIver & McIver (2006) demonstrated the existence of motion trapped modes in the coupled problem. In the frequency domain, the mathematical significance of a trapped mode is that, if a structure does not support a trapped mode at a particular frequency of oscillation, then the solution to a corresponding water-wave problem at that frequency is unique.

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A trapped-mode potential is a non-trivial solution of the homogeneous form of a frequency-domain water-wave problem obtained in the absence of any forcing. In the case of scattering and radiation problems this implies that the trapped-mode potential satisfies a homogeneous Neumann condition on the structure's surface, whereas for a coupled wave-structure motion problem the boundary condition on the structure contains both the potential and the structure's velocity but the equation of motion contains no forcing term. As demonstrated by McIver (1997), one consequence of the homogeneous Neumann condition for the particular class of sloshing trapped modes discovered by McIver (1996) is that the modes are orthogonal to any incident waves, and hence these trapped modes cannot be excited in a scattering problem in the time or frequency domains. However, the existence of a sloshing trapped mode of the type discovered by McIver (1996) implies the non-existence of the solution to a radiation problem at the trapped-mode frequency and this corresponds to a pole in the frequency-domain radiation potential at that frequency. As a consequence the sloshing trapped modes of McIver (1996) can be excited in the time-domain by an appropriate forced oscillation of a trapping structure (McIver, McIver & Zhang 2003). In contrast, McIver & McIver (2006) show that the existence of a motion trapped mode in the water-wave problem for a freely floating structure results in both the scattering and radiation potentials being well behaved at the trapped-mode frequency. Thus, motion trapped modes cannot be excited in the time domain either by the forced motion of the structure or, for a structure initially at rest in its equilibrium position, by incident waves. However, excitation of the motion trapped mode can be achieved by giving the structure a non-zero initial displacement or velocity. From a numerical perspective, the existence of the sloshing trapped modes discussed above has the consequence that it is difficult to accurately compute the added mass and damping coefficients near the trapped-mode frequency where these hydrodynamic coefficients are singular. However, for motion trapped modes all of the hydrodynamic coefficients are well behaved at the trapped mode frequency.

It was previously thought that sloshing trapped modes cannot persist when the trapping structure is allowed to float freely and respond to the hydrodynamic forces that act upon it. In general, a sloshing trapped mode exerts a non-zero hydrodynamic force on the structure which excites a motion of the structure that radiates energy to infinity, and hence damps the trapped mode. This is true for the particular class of sloshing trapping structures discovered by McIver (2006) whose properties are discussed above, but Motygin & Kuznetsov (1998) constructed two-dimensional sloshing trapping structures whose corresponding trapped modes exert no net force on the structure (although, perhaps, this was not recognized at the time). Such structures can still support the trapped modes when allowed to float freely as the fluid oscillation of the mode excites no motion of the structure. These last structures may be considered to be both sloshing and motion trapping structures and, henceforth, they will be referred to as passive trapping structures and the corresponding persistent oscillations as passive trapped modes. In this paper set notation is used to distinguish between the three classes of trapped modes. Let $\mathscr{S}$ be the set of all possible sloshing trapped modes and let $\mathscr{M}$ be the set of all possible motion trapped modes; the set of passive trapped modes $\mathscr{P} = \mathscr{S} \cap \mathscr{M}$. The original class of sloshing trapping modes investigated by McIver (1996) is in the set $\mathscr{S} \backslash \mathscr{M}$.

The purpose of the present work is to investigate the conditions required for the existence of passive trapped modes in two- and three-dimensional water-wave problems involving structures constrained to be free to move in the heave mode only, and to confirm the existence of such modes using time-domain simulations. The hydrodynamic properties of the trapping structures and time-domain excitation methods for the passive modes will also be discussed. The two-dimensional passive trapping structures are constructed by the method of Motygin & Kuznetsov (1998) and this is extended to three dimensions using ring singularities related to those used by McIver & McIver (1997); both methods have their origins in the inverse procedure used by Kyozuka & Yoshida (1981) to obtain wave-free oscillating structures. In the inverse procedure a potential is constructed that satisfies all the conditions of the problem with the exception of the boundary condition on the structure, and then appropriate streamlines that may be used as the surface of a structure are sought. Trapped modes that exert no net vertical hydrodynamic force on the corresponding structure may be obtained by using a specific arrangement of wave dipoles in the construction procedure. For example, for two-dimensional motion in water of infinite depth it is shown here that there is no force on the structure provided that the potential used in the inverse method decays faster than a dipole at infinity. An appropriate potential consists of a pair of horizontal wave dipoles in the free surface positioned to eliminate waves at infinity, and with opposite orientation to ensure that the dipole coefficient at infinity is zero.

A structure that supports a passive trapped mode at a particular frequency in a single mode of oscillation will not experience a net force nor move as a result of the persistent trapped mode oscillations of the surrounding fluid and consequently, the excitation methods used previously for sloshing trapped modes and motion trapped modes in the classes $\mathscr{S}\backslash\mathscr{P}$ and $\mathscr{M}\backslash\mathscr{P}$, respectively, are ineffective. However, an alternative excitation method in which the fluid is given an oscillatory pressure forcing on the free surface (McIver 1997) does prove effective. This is demonstrated analytically by using Green's theorem to show that the frequency-domain solution of the 'pressure potential' (the potential satisfying the freely floating body equations while undergoing the free-surface pressure forcing) does not exist at the trapped mode frequency. Another method for excitation of a passive trapped mode is the imposition of an initial free-surface elevation on the fluid surrounding the structure. However, we choose to use a pressure forcing here as a straightforward frequency-domain analysis readily yields both a necessary condition for excitation and an asymptotic result in time that provides the opportunity for comparisons with computations in the time domain.

In this paper the motion of a structure in heave only is considered in both two- and three-dimensional fluid domains. The problem of a moored floating structure free to respond to incident waves is formulated in § 2. The zero-force condition for a passive trapped mode is introduced in the context of the conditions for a motion trapping structure in § 3. The construction of the passive trapping structures in two and three dimensions is demonstrated in § 4, and in § 5 excitation methods are described and time domain simulations are presented to confirm the existence and illustrate the properties of passive trapped modes.

2. Formulation

Consider a moored surface-piercing structure, constrained to move in heave only, that is floating in an inviscid and incompressible fluid that may be of infinite or finite depth. The fluid is also of infinite extent in all horizontal directions and Cartesian coordinates (x, y, z) are chosen with the z-axis directed vertically upwards from the mean free surface. In the two-dimensional problems considered, the structure is assumed to extend indefinitely in the y direction so that attention is restricted to the x–z plane only. The vertical displacement and velocity of the structure are denoted by $Z(t)$ and $\dot{Z}(t)$ respectively, and the wetted surface of the structure in both two and three dimensions is denoted by Γ.

For a structure moored by an arrangement of linear springs and dampers the equation describing the motion of the structure is

$$M\ddot{Z}(t) = -\rho \int_{\Gamma} \frac{\partial \Phi}{\partial t} n_z \, \mathrm{d}S - \left[(\rho g W + \kappa) Z(t) + \gamma \dot{Z}(t)\right], \tag{2.1}$$

where M is the mass of the structure, W the water-plane area, ρ the density of the water and g the acceleration due to gravity. The constants κ and γ describe respectively the properties of the springs and dampers in the mooring system. The z-component of the inward normal to the structure is denoted n_z and $\Phi(\boldsymbol{x}, z, t)$ is the time-domain velocity potential describing the fluid motion, where $\boldsymbol{x}$ denotes the coordinates in the horizontal plane. The motion is subject to the initial conditions

$$\Phi(\boldsymbol{x}, z, 0) = 0, \quad \frac{\partial \Phi}{\partial t}(\boldsymbol{x}, z, 0) = 0, \tag{2.2}$$

so that the fluid is initially at rest (in the vicinity of the structure at least), and for all time

$$\nabla \Phi \to 0 \quad \text{as} \quad |\boldsymbol{x}| \to \infty. \tag{2.3}$$

The initial displacement $Z(0)$ and velocity $\dot{Z}(0)$ of the structure must also be prescribed.

The frequency-domain potential $\phi(\boldsymbol{x}, z, \omega)$ is obtained using the Fourier transform and, for fluid that is at rest for $t < 0$,

$$\phi(\boldsymbol{x}, z, \omega) = \int_0^{\infty} \Phi(\boldsymbol{x}, z, t) \, \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}t, \quad \operatorname{Im} \omega > 0. \tag{2.4}$$

The condition on the imaginary part of ω is required as in the fluid motions to be considered here $\Phi(\boldsymbol{x}, z, t)$ may not approach zero as $t \to \infty$. More specifically, in some circumstances a trapped mode is excited so that $\Phi(\boldsymbol{x}, z, t)$ may represent a bounded oscillation as $t \to \infty$ or, in a resonant situation, $\Phi(\boldsymbol{x}, z, t)$ may grow algebraically as $t \to \infty$. The potential $\phi(\boldsymbol{x}, z, \omega)$, as defined in (2.4), can be analytically continued on to the real ω axis (with the exception of any singular points that might arise, for example, from the existence of a trapped mode). It satisfies the usual frequency-domain equations governing the fluid motion which include Laplace's equation, the free-surface condition, the boundary condition on the structure, and an appropriate radiation condition. Fourier transformation of (2.1) and continuation on to the real axis gives the corresponding frequency-domain equation of motion

$$\begin{aligned}\left[\rho g W + \kappa - \mathrm{i}\omega\gamma - \omega^2 M\right] v(\omega) = -\mathrm{i}\omega\rho \int_{\Gamma} &[\mathrm{i}\omega\phi(\boldsymbol{x}, z, \omega) \\ &+ \Phi(\boldsymbol{x}, z, 0^+)] \, n_z \, \mathrm{d}S - (\rho g W + \kappa) Z(0) - \mathrm{i}\omega M \dot{Z}(0),\end{aligned} \tag{2.5}$$

where $v(\omega)$ is the Fourier transform of the velocity $\dot{Z}(t)$. The inversion formula for the potential is

$$\Phi(\boldsymbol{x}, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(\boldsymbol{x}, z, \omega) \, \mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{d}\omega = \frac{1}{\pi} \operatorname{Re} \int_0^{\infty} \phi(\boldsymbol{x}, z, \omega) \, \mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{d}\omega, \tag{2.6}$$

where the path of integration passes over any poles on the positive real ω axis, and this is used later to obtain an asymptotic expression for the fluid motion in the long-time limit. The reduction to a semi-infinite interval follows from $\phi(\boldsymbol{x}, z, -\omega) = \overline{\phi(\boldsymbol{x}, z, \omega)}$, $\omega \in \mathbb{R}$, and consideration of the poles.

3. Conditions for the existence of passive trapped modes

Trapped modes were first observed in uniqueness investigations into water-wave problems and it is in this context that the conditions for the existence of passive trapped modes are introduced. Consider two solutions $\{\phi_1, v_1\}$, $\{\phi_2, v_2\}$ to the frequency-domain water-wave problem. The difference potential $\phi = \phi_1 - \phi_2$ and velocity $V = v_1 - v_2$ satisfy a homogeneous water-wave problem. That is, the potential satisfies Laplace's equation

$$\nabla^2 \phi = 0 \tag{3.1}$$

within the fluid, the boundary condition

$$\frac{\partial \phi}{\partial z} = \frac{\omega^2}{g}\phi \quad \text{on} \quad F, \tag{3.2}$$

the free surface and the no-flow boundary condition on the surface of the structure

$$\frac{\partial \phi}{\partial n} = V n_z \quad \text{on} \quad \Gamma, \tag{3.3}$$

where V satisfies the homogeneous form of the equation of motion (2.5), that is,

$$[\rho g W + \kappa - \omega^2\{M + \mathrm{i}\gamma/\omega\}]V(\omega) = \omega^2 \rho \int_\Gamma \phi\, n_z \,\mathrm{d}S \tag{3.4}$$

(the terms involving the initial conditions cancel as they are identical for both solutions). The right-hand side of (3.4) is proportional to the hydrodynamic force on the structure due to the fluid oscillations. For a moored structure, the possibility of finite-energy motions that satisfy (3.3) and (3.4) with non-zero ϕ and V has been known for some time and there have been a number of recent developments (Evans & Porter 2007; Newman 2008). A necessary condition for the construction of such modes is that the damping constant γ is zero. The existence of non-trivial solutions with finite energy of (3.1)–(3.4) for structures without moorings, and with $V \neq 0$, has been established using the inverse method by McIver & McIver (2006, 2007), and solutions for half-immersed circular cylinders have been found by Porter & Evans (2009).

In addition to the motion trapped modes for which both the potential ϕ and the velocity V are non-zero, there is the possibility that finite-energy solutions to (3.1)–(3.4) exist with $\phi \neq 0$ but with $V = 0$ provided

$$\int_\Gamma \phi\, n_z \,\mathrm{d}S = 0 \tag{3.5}$$

(in which case it is permissible for γ to be non-zero). A solution of this type corresponds to a fluid motion that satisfies the homogeneous boundary condition for sloshing trapped modes

$$\frac{\partial \phi}{\partial n} = 0 \qquad \text{on} \quad \Gamma, \tag{3.6}$$

but for which the structure is able to float freely as the net hydrodynamic force on the structure due to the fluid motion is zero. Structures that support modes of this type will be here referred to as passive trapping structures. Insight into how such structures might be found can be gained from examining the conditions under which (3.5) may be satisfied. For the particular case of two-dimensional motions in infinite depth, any potential that corresponds to a finite-energy motion and that is wave free at infinity (and hence a trapped-mode potential $\phi_0(\boldsymbol{x}, z)$) satisfies

$$\phi_0 = \frac{\mathscr{D}\cos\theta}{R} + o\left(R^{-1}\right) \text{ as } R \to \infty, \tag{3.7}$$

where $R = \sqrt{x^2 + z^2}$ and θ are polar coordinates with θ measured from the downward vertical (see Ursell 1968, and the papers referenced therein). The constant $\mathscr{D}$ is referred to here as the dipole coefficient and, in general, to leading order the trapped mode is dipole-like at infinity. The trapped-mode potential also satisfies the homogeneous Neumann condition (3.6) on the structure and condition (3.2) on the free surface. Consider the fluid domain D bounded by the free-surface, the surface Γ of the structure, and an enclosing semicircle S_∞ at infinity in $z < 0$. An application of Green's theorem over D to the trapped-mode potential ϕ_0 and $u_0 = z + g/\omega_0^2$, where ω_0 is the trapped-mode frequency, gives

$$\int_\Gamma \phi_0 n_z \,\mathrm{d}S = \pi\mathscr{D} \tag{3.8}$$

(note that this result requires $\partial\phi_0/\partial n = 0$ on Γ). Thus, (3.5) is satisfied if and only if $\mathscr{D} = 0$, that is the coefficient of the dipole in the far-field expansion of ϕ_0 is zero. When the velocity $V = 0$ the equations for ϕ_0 are the same as those for a sloshing trapped mode, and it so happens that such modes with $\mathscr{D} = 0$ have been constructed previously by Motygin & Kuznetsov (1998).

For two-dimensional motion in fluid of constant finite depth h, Green's theorem yields

$$\int_\Gamma \phi_0 n_z \,\mathrm{d}S = -\int_{-\infty}^{\infty} \phi_0|_{z=-h} \,\mathrm{d}x \tag{3.9}$$

so that a necessary condition for the existence of a passive trapped mode at the frequency ω_0 is

$$\int_{-\infty}^{\infty} \phi_0|_{z=-h} \,\mathrm{d}x = 0. \tag{3.10}$$

In three dimensions, the derivations of the necessary conditions for the existence of passive trapped modes in a fluid infinite or constant finite depth h are almost identical to those for two dimensions. For a vertically axisymmetric three-dimensional structure in a fluid of infinite depth, any potential that is wave free at infinity satisfies (see Hulme 1982)

$$\phi_0 = \frac{\mathscr{D}\cos\theta}{R^2} + o\left(R^{-2}\right) \text{ as } R \to \infty, \tag{3.11}$$

where in this case $R = \sqrt{x^2 + y^2 + z^2}$ and θ is measured from the downward vertical. The application of Green's theorem to ϕ_0 and $u_0 = z + g/\omega_0^2$ on the three-dimensional fluid domain bounded by the free surface, the surface of the structure and an enclosing hemisphere in the lower half of the plane gives

$$\int_\Gamma \phi_0 n_z \,\mathrm{d}S = 2\pi\mathscr{D} \tag{3.12}$$

and, again, to obtain a passive trapped mode the dipole coefficient $\mathscr{D}$ for the potential must be zero. For a fluid of constant finite depth, the existence condition is the three-dimensional equivalent of the condition (3.10) for two dimensions, i.e.

$$\int_0^\infty \phi_0|_{z=-h}\, r\, \mathrm{d}r = 0, \tag{3.13}$$

where $r=\sqrt{x^2+y^2}$ is a horizontal polar coordinate.

4. Construction of passive trapping structures

4.1. *Two-dimensional structures*

Passive trapping structures are constructed here using an inverse procedure similar to that employed by McIver (1996) to construct the original class of sloshing trapping structures in the set $\mathscr{S}\backslash\mathscr{M}$. In that paper a potential satisfying the wave-free condition at infinity is constructed from two free-surface sources and the streamlines isolating the singularities from infinity are interpreted as the body contours. Thus, the singularities of the potential are not on the boundary of the fluid domain because the source points are within the bodies. For two-dimensional passive trapping structures in water of infinite depth the construction uses a pair of horizontal wave dipoles, as in Motygin & Kuznetsov (1998), and the trapped-mode potential is (correcting a sign)

$$\phi_0(x,z) = \frac{1}{K}\left[\frac{x-\xi}{(x-\xi)^2+z^2} - \frac{x+\xi}{(x+\xi)^2+z^2}\right] + ⨍_0^\infty \mathrm{e}^{uz}\,\frac{\sin u(x-\xi)-\sin u(x+\xi)}{u-K}\,\mathrm{d}u. \tag{4.1}$$

Here $K=\omega_0^2/g$ is the infinite-depth wavenumber corresponding to a trapped mode of frequency ω_0 and, to eliminate waves at infinity the dipoles are located symmetrically about the origin at $(\pm\xi,0)$ where $K\xi=n\pi$ with $n\in\mathbb{Z}^+$. Some properties of this potential are discussed in detail by Kuznetsov, Maz'ya & Vainberg (2002, section 4.2.2.3). A straightforward asymptotic analysis (see Appendix B) shows that this combination of equal-strength dipoles has a vanishing far-field dipole coefficient which, as explained in the previous section, is required for a passive trapped mode. The corresponding stream function, needed for the location of the streamlines, is

$$\psi_0(x,z) = -\frac{1}{K}\left[\frac{z}{(x-\xi)^2+z^2} - \frac{z}{(x+\xi)^2+z^2}\right] + ⨍_0^\infty \mathrm{e}^{uz}\,\frac{\cos u(x-\xi)-\cos u(x+\xi)}{u-K}\,\mathrm{d}u, \tag{4.2}$$

where the arbitrary constant of integration has been chosen to ensure decay to zero at infinity. In the inverse procedure the stream function ψ_0 is used to construct the trapping structures for a particular frequency ω_0 by choosing values of δ in the equation $\psi_0(x,z;K)=\delta$ and some typical results are shown in figure 1 for the case $K\xi=\pi$.

The same approach can be adopted for a finite-depth fluid domain and the appropriate combination of horizontal wave dipoles gives a trapped-mode potential

$$\phi_0(x,z) = 2⨍_0^\infty \frac{u(\sin u(x-\xi)-\sin u(x+\xi))\,\mathrm{e}^{uz}(1+\mathrm{e}^{-2u(z+h)})}{u-K-(u+K)\,\mathrm{e}^{-2uh}}\,\mathrm{d}u \tag{4.3}$$

(the potential for a single dipole can be obtained, after a little calculation, from equations (B 56) and (B 57) of Linton & McIver 2001) and a corresponding stream function

$$\psi_0(x,z) = 2⨍_0^\infty \frac{u(\cos u(x-\xi)-\cos u(x+\xi))\,\mathrm{e}^{uz}(1-\mathrm{e}^{-2u(z+h)})}{u-K-(u+K)\,\mathrm{e}^{-2uh}}\,\mathrm{d}u. \tag{4.4}$$

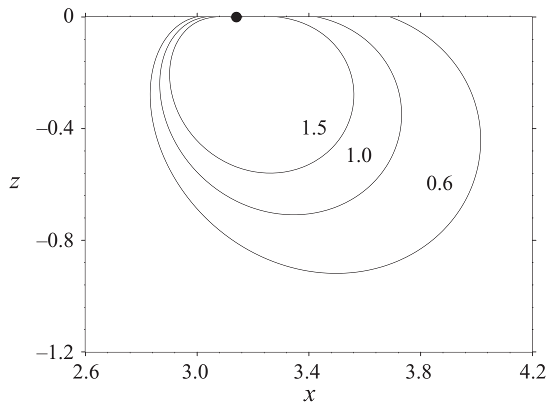

FIGURE 1. The right-hand element of passive trapping structures obtained from the stream function (4.2) with $\xi = \pi$, $K = 1$. Each streamline is marked with the corresponding value of δ.

To eliminate waves at infinity it is required that the finite-depth wavenumber k satisfies $k\xi = n\pi$, $n \in \mathbb{Z}^+$, and then the corresponding frequency parameter $K = k \tanh kh$. The geometries of the resulting trapping structures are qualitatively similar to those shown in figure 1. The excitation of passive trapped modes in a fluid of constant finite depth will be demonstrated later in the paper and the finite-depth passive trapping structures are used in these simulations. The condition (3.10) for a zero hydrodynamic force on the passive trapping structure in finite depth is verified in Appendix C.

Passive trapping structures support both sloshing and motion trapping modes because (3.3), (3.4) and (3.6) are all satisfied simultaneously by the (non-trivial) potential ϕ_0. Despite the fact that the structures also support sloshing trapped modes, for passive trapping structures both the radiation and scattering potentials exist at the trapped-mode frequency. Thus, in particular, the added mass and damping coefficient do not exhibit the singular behaviour at the trapped-mode frequency of the trapping structures corresponding to trapped modes in $\mathscr{S}\backslash\mathscr{M}$. To obtain numerical verification of these properties, a standard boundary-element method (BEM) frequency-domain code was used to compute the non-dimensional added mass μ and damping ν for a passive and sloshing trapping structure, within the classes $\mathscr{P}$ and $\mathscr{S}\backslash\mathscr{M}$ respectively, around the trapped-mode wavenumber $K = 1$. (The particular passive trapping structure used in these calculations corresponds to the streamline $\delta = 1.5$ shown in figure 1.) For the passive trapping structure, the errors inherent in a numerical approximation manifest themselves in very localized variations in μ and ν around the trapped-mode frequency as may be seen in figure 2; these are present because the discretization of the trapping structure is actually a near-trapping structure with the corresponding radiation potential possessing a complex resonance located very close to the real-ω axis. As the number of panels n_p is increased the peaks reduce in width, increase in height, and converge on $K = 1$. On the other hand, the singular behaviour of the added mass coefficient for the $\mathscr{S}\backslash\mathscr{M}$ trapping structure shown in figure 3 is much more significant than that for the passive trapping structure. The added mass tends to positive and negative infinity as K approaches the asymptote $K = 1$ from the left and the right respectively with the asymptotic behaviour occurring over a much larger range of K values than in the passive trapping structure case. Furthermore, the damping coefficient is close to zero for most of the range of K over which the added mass behaves in a singular manner; the passive trapping

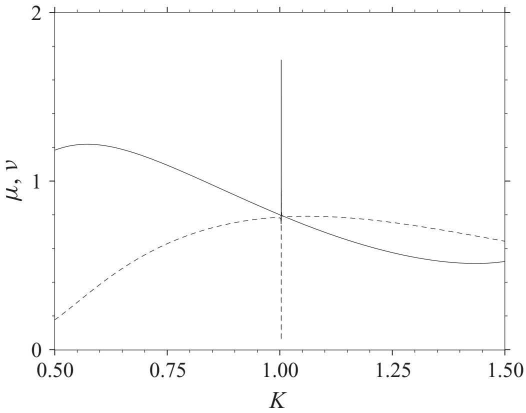

FIGURE 2. Numerical calculations of the non-dimensional added mass μ (———) and damping ν (– – –) as a function of the frequency parameter K for the structure $\delta = 1.5$ (see figure 1) discretized with 402 panels.

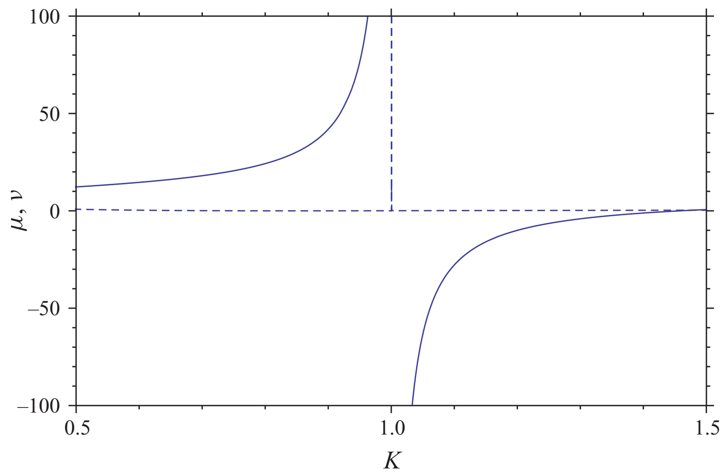

FIGURE 3. (Colour online) Numerical calculations of the non-dimensional added mass μ (———) and damping ν (– – –) around the trapped-mode frequency corresponding to $K = 1$ for a sloshing trapping structure of the class $\mathscr{S}\backslash\mathscr{M}$.

structure has finite non-zero values for μ and ν near $K = 1$. These features are robust under changes in the number of panels used. Therefore, although passive trapped modes obey the same equations as sloshing trapped modes in the class $\mathscr{S}\backslash\mathscr{M}$, the hydrodynamic properties of the corresponding structures are significantly different.

4.2. *Three-dimensional structures*

The three-dimensional passive trapping structures are assumed to have a vertical axis of symmetry and are constructed from ring-dipole potentials. Previously, McIver & McIver (1997) used an axisymmetric ring-source potential

$$\phi_s(r, z) = 4\pi^2 \mathrm{i} K c\, \mathrm{e}^{Kz}\, J_0(Kr_<) H_0(Kr_>) + 8c \int_0^\infty (u \cos uz + K \sin uz) I_0(ur_<) K_0(ur_>) \frac{u}{u^2 + K^2}\, \mathrm{d}u \quad (4.5)$$

to generate vertically axisymmetric sloshing trapping structures in infinite depth, where J_0, I_0, K_0 and H_0 denote standard Bessel, modified Bessel and Hankel functions of order zero and where $r_< = \min\{r, c\}$ and $r_> = \max\{r, c\}$. To eliminate the first term in (4.5) corresponding to the outgoing radial waves the radius c of the ring source is chosen to satisfy

$$J_0(Kc) = 0, \tag{4.6}$$

where $K = \omega_0^2/g$ and ω_0 is the trapped-mode frequency.

To construct passive trapping structures, the potential due to a ring-dipole in the free-surface is used and again the radius of the ring chosen to eliminate the far-field waves. The ring-dipole is obtained from (4.5) by differentiation with respect to the ring radius c (the multiplicative factor c is first removed from (4.5) so that no source-like terms remain after differentiation); the resulting ring-dipole potential is

$$\phi_0(r, z) = -4\pi^2 \mathrm{i} K^2 \,\mathrm{e}^{Kz}\, J_0(Kr)H_1(Kc) - 8\int_0^\infty (u\cos uz + K\sin uz)\, I_0(ur)K_1(uc)\frac{u^2}{u^2+K^2}\,\mathrm{d}u, \quad 0 \leqslant r < c, \tag{4.7}$$

$$\phi_0(r, z) = -4\pi^2 \mathrm{i} K^2 \,\mathrm{e}^{Kz} J_1(Kc)H_0(Kr) - 8\int_0^\infty (u\cos uz + K\sin uz)\, I_1(uc)K_0(ur)\frac{u^2}{u^2+K^2}\,\mathrm{d}u, \quad r > c, \tag{4.8}$$

and to ensure that it is wave-free at infinity the coefficient of $H_0(Kr)$ in (4.8) is set to zero, so that

$$J_1(Kc) = 0. \tag{4.9}$$

Thus $Kc = j_{1,i}$ where $j_{1,i}$, $i = 1, 2, \ldots$, are the positive zeros of the Bessel function of order one arranged in ascending order. To represent a passive trapped mode the potential must also have a zero dipole coefficient at infinity (see § 3) and this is verified in Appendix D. The Stokes stream function for the above potential is

$$\psi_0(r, z) = -4\pi^2 r K^2 \,\mathrm{e}^{-Ky}\, J_1(Kr)Y_1(Kc) - 8r\int_0^\infty (u\sin uz - K\cos uz) I_1(ur)K_1(uc)\frac{u^2}{u^2+K^2}\,\mathrm{d}u, \quad 0 \leqslant r < c, \tag{4.10}$$

$$\psi_0(r, z) = -8r\int_0^\infty (u\sin uz - K\cos uz) I_1(uc)K_1(ur)\frac{u^2}{u^2+K^2}\,\mathrm{d}u, \quad r > c, \tag{4.11}$$

where the constants of integration are chosen to give $\psi \to 0$ as $r \to \infty$. Plots of typical stream surfaces for this deep-water ring-dipole stream function are shown in figure 4 for the ring dipole radius of $c = j_{1,1}/K$ with the infinite-depth wavenumber chosen to be $K = 1$. The corresponding structure is generated by the rotation of the contours about the z-axis and therefore the structures are toroidal and enclose a portion of the free-surface. Numerical calculations (performed with the panel code WAMIT) of the added mass and damping for one of the structures in figure 4 are given in figure 5 and, as in the two-dimensional case, there is a very localized lack of smoothness around the trapped-mode frequency due to numerical errors.

Finite-depth passive trapping structures are constructed in a similar fashion. A suitable dipole potential is $\phi_0 = \partial(R_0/c)/\partial c$ where

$$R_0(r, z; c, 0) = 4\pi c \int_0^\infty \frac{u\cosh u(h+z)}{u\sinh uh - K\cosh uh} J_0(ur)J_0(uc)\,\mathrm{d}u \tag{4.12}$$

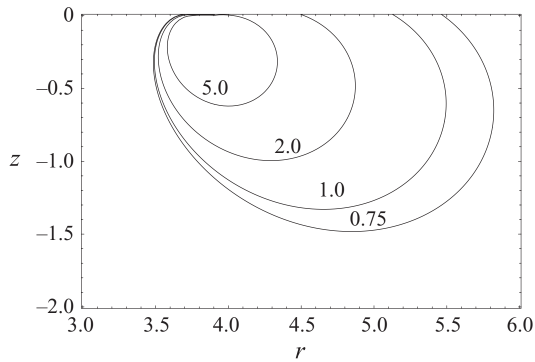

FIGURE 4. The trace of passive trapping structures in the r–z plane obtained from the solution of $\psi_0(r, z) = \delta$ for $c = j_{1,1}/K$ and with $K = 1$. Each streamline is marked with the corresponding value of δ.

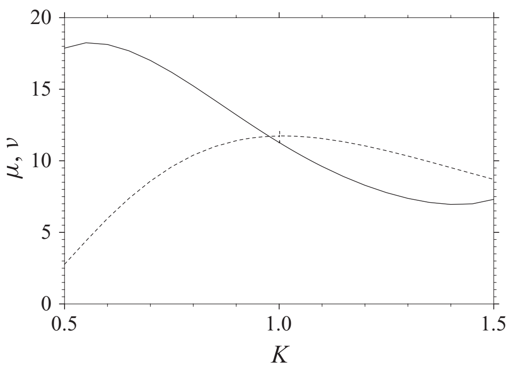

FIGURE 5. The non-dimensional added mass μ (———) and damping ν (– – –) as a function of frequency parameter K for the structure corresponding to the streamline value $\delta = 5.0$ in figure 4 discretized with 2304 panels.

is a finite-depth ring-source potential (Hulme 1983), where $K = k \tanh kh$ and k is the finite-depth wavenumber. With the ring radius chosen to be $c = j_{1,i}/k$ for $i = 1, 2, \ldots$, the singularity in the denominator of the integrand is cancelled by the zero of the Bessel function $J_1(uc)$ at $u = k$ and the integral representation of the trapped-mode potential is

$$\phi_0(r, z) = -4\pi \int_0^\infty \frac{u^2 \cosh u(h + z)}{u \sinh uh - K \cosh uh} J_0(ur) J_1(uc)\, \mathrm{d}u \tag{4.13}$$

with no radiated wave term present. The corresponding stream function is

$$\psi_0(r, z) = -4\pi r \int_0^\infty \frac{u^2 \sinh u(h + z)}{u \sinh uh - K \cosh uh} J_1(ur) J_1(uc)\, \mathrm{d}u. \tag{4.14}$$

The stream surfaces that correspond to passive trapping structures are qualitatively similar to those in the infinite-depth case. The finite-depth passive trapping structures are used here for the numerical demonstration of the existence of passive trapped modes. To confirm that the stream function (4.14) corresponds to a passive trapped mode, the potential (4.12) must satisfy the finite-depth condition (3.13). A numerical approximation to the infinite integral can be obtained by replacing the upper limit of the integral by a large, but finite, radius and values for the integral of the order of 10^{-6} were obtained for a domain of radius $20h$.

5. Excitation of passive trapped modes

5.1. *Excitation condition in the frequency domain*

If a frequency-domain potential possesses a simple pole at a real frequency ω then the corresponding time-domain potential describes a persistent oscillation because the residue at the pole yields a term proportional to $e^{-i\omega t}$ when the inverse Fourier transform is evaluated, while a double pole leads to the unbounded growth of an oscillation in the time domain. Thus, the non-existence of a frequency-domain potential at a certain real frequency implies that trapped-mode excitation is possible in a corresponding time-domain problem. For example, the non-existence of a radiation potential at the trapped-mode frequency in the forced motion problem involving a sloshing trapping structure of the class $\mathscr{S}\backslash\mathscr{M}$ means that corresponding forced motions of the structure will excite the trapped mode (see McIver *et. al.* 2003 for details). However, for all motion trapping structures (including passive trapping structures) the radiation potentials are well behaved at the trapped-mode frequency and the trapped modes cannot be excited by forced motions of the structure. Excitation of motion trapped modes of the class $\mathscr{M}\backslash\mathscr{S}$ can be achieved by giving the structure an initial displacement or velocity because, in general, there is a singularity in the frequency-domain velocity $v(\omega)$. In the absence of moorings, this follows from the resonance condition

$$\rho g W - \omega^2\left[M + a(\omega) + \mathrm{i}b(\omega)/\omega\right] = 0, \tag{5.1}$$

which is a necessary condition for the existence of a non-trivial solution for $v(\omega)$ of the homogeneous form of the equation of motion for a floating structure, (A 5) in Appendix A (see also McIver & McIver 2006). However, for passive trapped modes there is, in general, no pole in $v(\omega)$ on the real ω axis and, consequently, no excitation can arise from the imposition of an initial displacement or velocity and an alternative excitation method must be used to obtain a persistent passive mode of oscillation in the fluid.

Using Green's theorem, McIver (1997) demonstrates explicitly how the violation of the existence condition for a frequency-domain potential due to a pressure forcing on the free surface corresponds to the unbounded growth of a trapped-mode oscillation in the time domain. A similar approach can be taken for passive trapped modes by considering a potential due to an oscillatory forcing pressure in the presence of a freely floating passive trapping structure. In this case, the pressure forcing potential ϕ_p satisfies the boundary condition on the structure (3.3) and the equation of motion (3.4) as well as a modified free-surface boundary condition

$$\frac{\partial \phi_p}{\partial z} - K\phi_p = f \quad \text{on} \quad F, \tag{5.2}$$

where f is proportional to the pressure on the free surface. The passive trapped-mode potential ϕ_0 satisfies the homogeneous free-surface condition, the homogeneous Neumann condition on the structure surface (3.6), and the zero-force condition (3.5). Therefore, under the assumption that ϕ_p exists, an application of Green's theorem to ϕ_p and ϕ_0 at the trapped-mode frequency $\omega = \omega_0$ over the fluid domain yields the condition

$$\int_F \phi_0 f \, \mathrm{d}S = 0, \tag{5.3}$$

because the integrals over the structure's surface Γ, the far-field control surface S_∞ and the bottom boundary all vanish. If the condition is not satisfied then ϕ_p does not exist at the trapped-mode frequency and it will be possible to excite a persistent oscillation corresponding to the trapped mode in the time domain by application of the corresponding time-dependent forcing pressure. Note that the analysis is valid in both two and three dimensions.

Many different forcing pressure profiles f can be used to excite passive trapped modes. Possible choices are $f(\boldsymbol{x}) = \phi_0(\boldsymbol{x}, 0)$, where $\boldsymbol{x}$ denotes the coordinates in the horizontal plane, and any forcing function that has the same sign as $\phi_0(\boldsymbol{x}, 0)$ everywhere on the free surface. It is also possible to choose a pressure profile such that the existence condition is satisfied and so that no excitation occurs. For example, this may be achieved in two dimensions using forcing functions of the form

$$f(x;\omega_0) = \begin{cases} f_0 + f_1 x + f_2 x^2, & x \in F_I, \\ 0, & x \in F_E, \end{cases} \tag{5.4}$$

where F_I (F_E) is the internal (external) free surface. By setting $f_1 = 0$, the forcing function becomes even and the existence condition then requires

$$f_0 = -f_2 \frac{\displaystyle\int_{F_I} \phi_0(x, 0) x^2 \, \mathrm{d}x}{\displaystyle\int_{F_I} \phi_0(x, 0) \, \mathrm{d}x}. \tag{5.5}$$

In particular, with $f_2 = 1$ this yields a forcing profile $f(x, \omega_0) = f_0 + x^2$ which does not excite the passive trapped mode. To illustrate the excitation analysis, time-domain simulations were performed with the various free-surface forcing pressures described above. These forcing pressures do not need to exist indefinitely and both persistent and transient forcing pressures are considered; the former results in resonant growth of the passive trapped oscillation while the latter yields a persistent steady fluid oscillation. Finally, the oscillatory pressure forcing which satisfies the existence condition is shown not to excite the passive trapped mode. In all simulations it is assumed that the structure floats freely so that no mooring forces are present.

5.2. *Resonant free-surface pressure forcing*

The application of a persistent oscillatory pressure forcing to the free-surface surrounding a passive trapping structure at the trapped-mode frequency ω_0 will generally result in a fluid oscillation of indefinitely increasing amplitude. This can be demonstrated by an analysis of the time-domain problem in the long-time limit which involves the asymptotics of the frequency-domain solution as $\omega \to \omega_0$. The application of a pressure forcing on the free-surface means that the time-domain potential Φ_p describing the response of the fluid to this forcing must satisfy the dynamic free-surface boundary condition

$$\frac{\partial \Phi_p}{\partial t} + g\eta = \frac{P(x, t)}{\rho}, \tag{5.6}$$

where P is the forcing which is assumed to have the form

$$P(x,t) = \frac{\rho g f(x)}{\omega_0} \sin \omega_0 t. \tag{5.7}$$

Provided the fluid motion starts from rest then Fourier transformation yields the frequency-domain free-surface condition

$$\left(K - \frac{\partial}{\partial z}\right)\phi_p = \frac{\mathrm{i}\omega}{\omega^2 - \omega_0^2} f(x) \tag{5.8}$$

for $\omega \neq \omega_0$, where ϕ_p is related to Φ_p by (2.4). An estimate of the long-time response of the fluid can be made by an application of the inverse Fourier transform (2.6) to a Laurent expansion in ω of the frequency-domain potential ϕ_p about ω_0. This expansion is determined in a similar way to that used by McIver *et. al.* (2003, Appendix A) for a heaving sloshing trapping structure and the leading order term is

$$\phi_p(x,z,\omega) = \frac{\mathrm{i}g\omega A\phi_0(x,z)}{(\omega^2 - \omega_0^2)^2} + O(1) \quad \text{as} \quad \omega \to \omega_0 \tag{5.9}$$

for a two-dimensional passive trapping structure in finite depth. The amplitude A is determined by an application of Green's theorem to ϕ_p and ϕ_0 for $\omega \simeq \omega_0$ and the subsequent substitution of (5.9) for ϕ_p. In contrast to the heave problem for a sloshing trapping structure, where the boundary condition on the structure is not homogeneous, the inhomogeneous term in the pressure-forcing problem is in the free-surface boundary condition and it may be shown that

$$A = \frac{\displaystyle\int_F \phi_0(\xi,0) f(\xi)\,\mathrm{d}\xi}{\displaystyle\int_F [\phi_0(\xi,0)]^2\,\mathrm{d}\xi}. \tag{5.10}$$

The numerical calculation of A is straightforward as both the numerator and the denominator can be evaluated from the trapped-mode potential and the pressure profile $f(x)$. Note that if the condition (5.3) holds then $A=0$ and no trapped mode is excited.

Given that the dominant pole structure of ϕ_p is known, the asymptotic form of Φ_p for large time can be found by calculating the required residues. The free-surface elevation η follows from (5.6) to give

$$\eta \sim -\frac{A\phi(x,0)}{2\omega_0} t \cos \omega_0 t \quad \text{as} \quad t \to \infty, \tag{5.11}$$

and the expected resonant growth in exhibited. Time-domain simulations display similar resonant behaviour and a comparison with the asymptotic result is given in figure 6. In these calculations, the non-dimensional trapped-mode frequency is $\Omega = \sqrt{4 \tanh 4}$ and the free-surface pressure profile has the form

$$f(x) = \begin{cases} x_0^2 - x^2, & x \in F_I, \\ 0, & x \in F_E, \end{cases} \tag{5.12}$$

where x_0 is chosen to satisfy $\phi_0(x_0,0)=0$ and $\operatorname{sgn} f(x) = \operatorname{sgn} \phi_0(x,0)$ on the internal free-surface F_I so that the existence condition (5.3) is violated. The free-surface elevation was evaluated at the mid-point between the structures and compared to the asymptotic amplitude prediction $A\phi(0,0)/2$ giving a good agreement. Note that

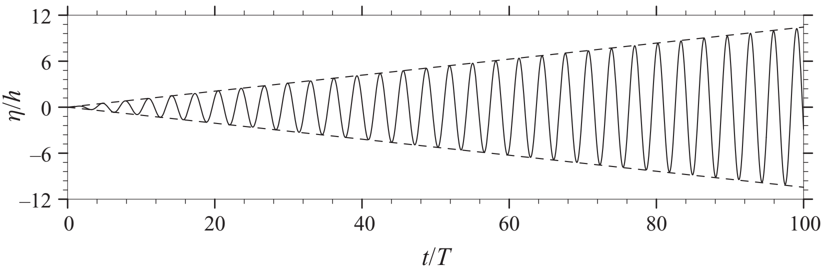

FIGURE 6. Time-domain simulations (——) of the free-surface elevation η/h at the mid-point between the elements of a passive trapping structure compared to the asymptotic prediction of the oscillation amplitude (– – –) when a forcing free-surface pressure is applied.

the time-domain simulations involve non-dimensional times t/T and distances x/h where $T=\sqrt{g/h}$ and the non-dimensional frequency is defined as $\Omega=\omega/T$.

5.3. *Time-domain simulations of transient excitations*

5.3.1. *Two dimensions*

Persistent passive trapped-mode oscillations with constant amplitude can be excited by applying transient pressure oscillations to the free surface. Here a pressure forcing

$$P(x,t)=t^3\,\mathrm{e}^{-t}\,\frac{\phi_0(x,0)}{\phi_0(0,0)} \tag{5.13}$$

is applied on the internal free surface; the factor $\phi_0(0,0)$ is chosen to ensure that the maximum magnitude of the forcing is of order unity. Time-domain results for the response of the fluid and the structure are shown in figure 7 to illustrate how a persistent motion of the free surface can occur without the excitation of a significant motion of the structure. In fact, the structure does move slightly due to the excitation of a motion resonance at the frequency $\Omega\approx 1.28$ that may be observed in the discrete Fourier transform of the structure's displacement shown in figure 8(*b*). A small peak at the trapped-mode frequency (denoted by its non-dimensional value 2.00) is also apparent implying that a small force is exerted on the structure by the passive trapped mode. However, this is to be expected because the discretized structure is an approximation to the passive trapping structure and so the trapped mode is actually a complex resonance with very small decay constant. The discrete Fourier transform of the free-surface elevation at the mid-point shown in figure 8(*a*) has a dominant peak at the trapped-mode frequency. It should be noted that other methods can be used to excite passive trapped modes, for example if an initial free-surface elevation is specified on the internal free surface that is non-zero and not orthogonal to the trapped mode then a persistent fluid oscillation will occur.

If the pressure profile $f(x)$ in (5.7) is chosen according to (5.4) and (5.5) so that the existence condition is satisfied then the passive trapped mode is not excited. This is demonstrated here in a time-domain simulation involving the same passive trapping structure as the previous simulations and with a free-surface pressure forcing of the form

$$P(x,t)=\frac{x^2+f_0}{\omega_0}\sin(\omega_0 t), \tag{5.14}$$

where f_0 was computed numerically according to (5.5) with $f_2=1$.

The results of this simulation are shown in figure 9 and it is clear that no resonant growth in the free-surface oscillation occurs due to this pressure forcing. The free

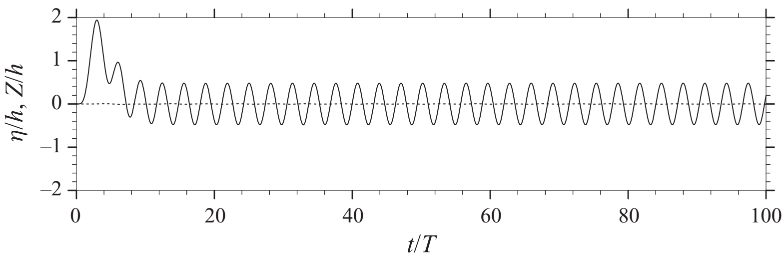

FIGURE 7. Displacements η/h of the mid-point of the free-surface (———) and Z/h of the structure (– – –) due to a transient pressure forcing.

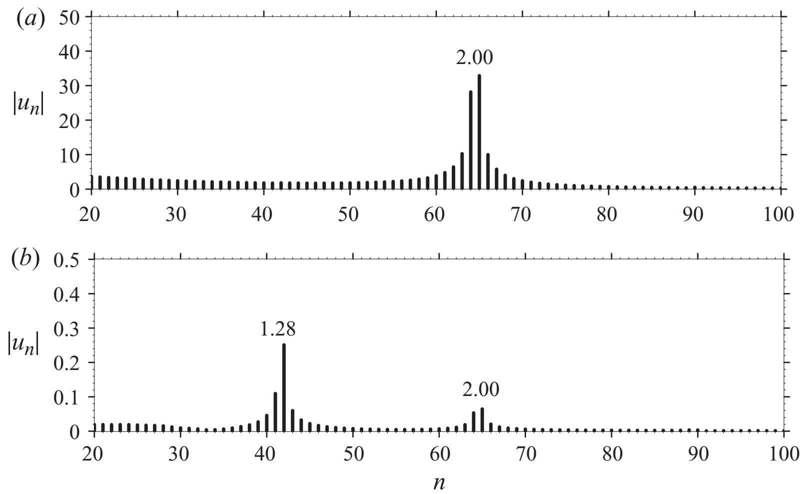

FIGURE 8. Discrete Fourier transforms of the displacements (*a*) η/h of the mid-point of the free surface and (*b*) Z/h of the structure, both of which are plotted in figure 7. Here $|u_n|$ is $N^{-1/2}$ times the amplitude of the Fourier component with index n, where N is the number of samples in the signal.

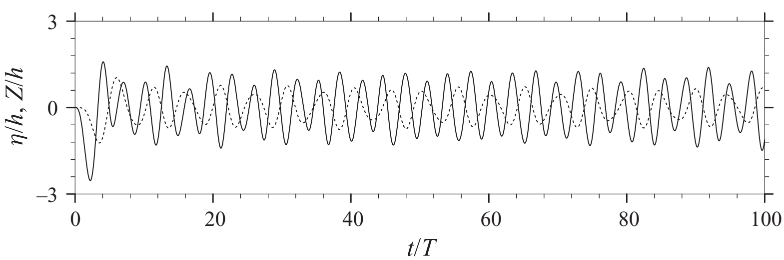

FIGURE 9. Displacement η/h of the mid-point of the free surface (———) and Z/h of the structure (– – –) due to the oscillatory pressure forcing (5.14).

surface does, however, oscillate at the frequency ω_0 due to the time-dependence of the oscillatory forcing (5.14) but the structure also undergoes appreciable oscillations at this frequency. Although the structure's oscillations of frequency ω_0 are smaller than those of the free surface, as can be seen from the Fourier transforms in figure 10, the two oscillations are of the same order of magnitude. In contrast to this, when the passive trapped mode is excited the motion of the structure at the frequency ω_0 is

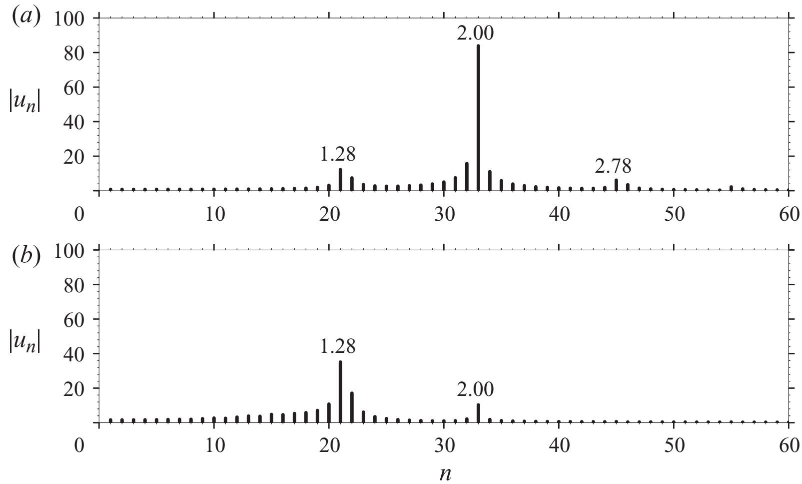

FIGURE 10. Discrete Fourier transforms of the displacements (a) η/h of the mid-point of the free surface and (b) Z/h of the structure, as shown in figure 9.

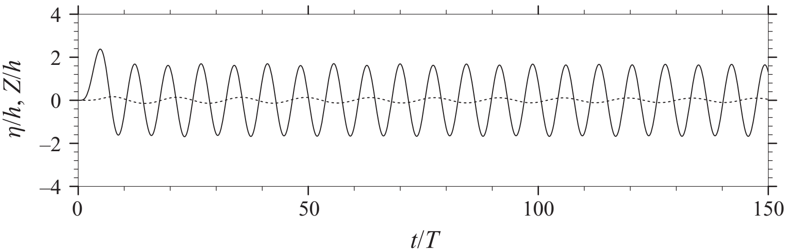

FIGURE 11. Displacements η/h of the mid-point $r=0$ of the free surface (——) and Z/h of the passive trapping structure (– – –) due to a transient pressure forcing in three dimensions.

small relative to the oscillations of the surrounding fluid. Thus, the frequency-domain analysis of the excitation of the passive trapped modes in § 5.1 is validated numerically in the time domain.

5.3.2. *Three dimensions*

To demonstrate excitation in three dimensions, a passive trapping structure was constructed from the three-dimensional finite-depth stream function (4.14) with the wavenumber $kh=1.0$ and the stream-surface constant $\delta=2.0$. A transient pressure forcing of the form (5.13) (with x replaced by r) was applied to the internal free surface and the resulting motion of the mid-point of the internal free surface and of the structure are illustrated in figure 11. After the initial transient has decayed a persistent oscillation of the free surface accompanied by a small oscillatory motion of the structure is observed. A discrete Fourier transform of the two signals reveals that a motion resonance of non-dimensional frequency $\Omega=0.46$ is excited in addition to the passive trapped mode of frequency $\Omega=\sqrt{1.0\tanh 1.0}\approx 0.87$. This resonance is observed as the only significant peak frequency in the Fourier transform of the structure's motion (figure 12) and as a minor peak in the transform of the free-surface oscillation at a frequency lower than the passive trapped-mode frequency. The persistent oscillation of the free surface occurs at the trapped-mode frequency

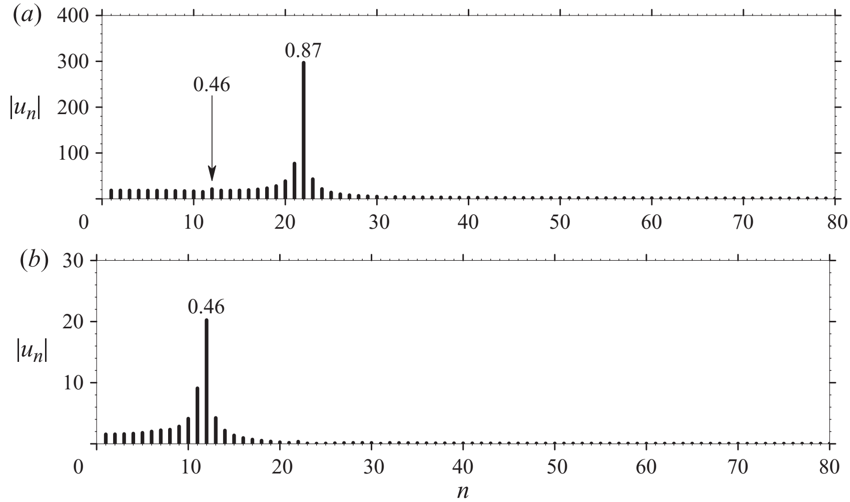

FIGURE 12. Discrete Fourier transforms of the displacements (*a*) η/h of the mid-point of the free surface and (*b*) Z/h of the structure, as shown in figure 11.

which confirms the existence of a passive trapped mode in an axisymmetric three-dimensional problem.

Appendix A. The frequency-domain equation of motion

In the frequency-domain equation of motion (2.5), the potential ϕ may be decomposed in terms of a scattering potential ϕ_S and a radiation potential ϕ_R so that

$$\phi(\boldsymbol{x}, z, \omega) = \phi_S(\boldsymbol{x}, z, \omega) + v(\omega)\phi_R(\boldsymbol{x}, z, \omega). \tag{A 1}$$

It is known that (Mei, Stiassnie & Yue 2005, § 8.12.2)

$$\phi_R(\boldsymbol{x}, z, \omega) \sim \Omega(\boldsymbol{x}, z) \neq 0 \quad \text{as} \quad \omega \to \infty \tag{A 2}$$

and thus by the convolution theorem, the inverse Fourier transform of (A 1) yields

$$\Phi(\boldsymbol{x}, z, t) = \Phi_S(\boldsymbol{x}, z, t) + \int_0^t \dot{X}(\tau)\Gamma_R(\boldsymbol{x}, z, t-\tau)\,\mathrm{d}\tau + \dot{X}(t)\Omega(\boldsymbol{x}, z), \tag{A 3}$$

where Φ_S is the time-domain scattering potential and Γ_R is the inverse Fourier transform of $\phi_R - \Omega$. It follows that

$$\Phi(\boldsymbol{x}, z, 0^+) = \Phi_S(\boldsymbol{x}, z, 0) + \dot{X}(t)\Omega(\boldsymbol{x}, z), \tag{A 4}$$

so that the equation of motion in the frequency domain can be written as

$$\begin{aligned}&[\rho g W + \kappa - \omega^2\{M + a(\omega) + \mathrm{i}(b(\omega) + \gamma)/\omega\}]v(\omega) \\ &\quad = -\mathrm{i}\omega[X(\omega) + \rho \int_\Gamma \Phi_S(\boldsymbol{x}, z, 0)\, n_z\,\mathrm{d}S + (M + a(\infty))\dot{Z}(0)] - (\rho g W + \kappa)Z(0),\end{aligned} \tag{A 5}$$

where the exciting force

$$X(\omega) = \mathrm{i}\omega\rho \int_\Gamma \phi_S(\boldsymbol{x}, z, \omega)\, n_z\,\mathrm{d}S \tag{A 6}$$

and the added mass and damping coefficients, respectively $a(\omega)$ and $b(\omega)$, are defined by

$$a(\omega) + \mathrm{i}b(\omega)/\omega = \rho \int_\Gamma \phi_R(\boldsymbol{x}, z, \omega)\, n_z \,\mathrm{d}S. \tag{A 7}$$

The coefficient

$$a(\infty) = \rho \int_\Gamma \Omega(\boldsymbol{x}, z)\, n_z \,\mathrm{d}S \tag{A 8}$$

may be formally identified with the added mass in the limit of infinite frequency (Mei *et al.* 2005, § 8.12); the term in $a(\infty)$ is mistakenly omitted from the corresponding equation in McIver & McIver (2006, equation (1)).

Appendix B. Asymptotic analysis of the two-dimensional potential

In this appendix it is shown that the potential given in (4.1) has no far-field dipole coefficient. In terms of polar coordinates defined by

$$x = R\sin\theta \quad \text{and} \quad z = -R\cos\theta, \tag{B 1}$$

so that θ is measured from the downward vertical, then

$$\frac{x-\xi}{(x-\xi)^2+z^2} = \frac{R\sin\theta - \xi}{R^2 - 2\xi R\sin\theta + \xi^2} = \frac{1}{\xi}\left[\frac{\xi\sin\theta}{R} + O\left(\xi^2/R^2\right)\right] \quad \text{as} \quad R/\xi \to \infty \tag{B 2}$$

and hence

$$\xi\left[\frac{x-\xi}{(x-\xi)^2+z^2} - \frac{x+\xi}{(x+\xi)^2+z^2}\right] = O\left(\xi^2/R^2\right) \quad \text{as} \quad R/\xi \to \infty. \tag{B 3}$$

The integral term

$$⨍_0^\infty \frac{\mathrm{e}^{uz}\sin u(x-\xi)}{u-K}\,\mathrm{d}u = -\operatorname{Im} ⨍_0^\infty \frac{\mathrm{e}^{-\mathrm{i}u(x-\xi+\mathrm{i}z)}}{u-K}\,\mathrm{d}u = -\operatorname{Im} ⨍_0^\infty \frac{\mathrm{e}^{-\mathrm{i}K(x-\xi+\mathrm{i}z)t}}{t-1}\,\mathrm{d}t \tag{B 4}$$

and the asymptotics of the last integral as $R \to \infty$ are discussed in Appendix A of Motygin & McIver (2003). In particular, it follows from (A 6) of that paper that as $KR \to \infty$:

$$⨍_0^\infty \frac{\mathrm{e}^{uz}\sin u(x-\xi)}{u-K}\,\mathrm{d}u = -\pi\,\mathrm{sgn}\,x\,\cos K(x-\xi)\,\mathrm{e}^{Kz} - \frac{\sin\theta}{KR} + O((KR)^{-2}), \tag{B 5}$$

and thus

$$\begin{aligned} ⨍_0^\infty \mathrm{e}^{uz}\frac{\sin u(x-\xi) - \sin u(x+\xi)}{u-K}\,\mathrm{d}u &= -2\pi\,\mathrm{sgn}\,x\,\sin Kx\,\sin K\xi\,\mathrm{e}^{Kz} + O((KR)^{-2}) \\ &= O((KR)^{-2}) \quad \text{as} \quad KR \to \infty, \end{aligned} \tag{B 6}$$

because $K\xi = n\pi$ with $n \in \mathbb{Z}^+$. From the above results it follows that the potential given in (4.1) is $O((KR)^{-2})$ as $KR \to \infty$ and hence has no dipole in its far-field expansion.

Appendix C. Verification of a condition for zero force

Here it is shown that the potential ϕ_0 defined in (4.3) satisfies the condition (3.10) which, for two-dimensional motions in finite-depth water, is necessary for the hydrodynamic force on the structure to be zero. First of all, it can be noted that

$$\phi_0(x, z) = \phi_1(x - \xi, z) - \phi_1(x + \xi, z) \tag{C 1}$$

where $\phi_1(x, z)$ is a horizontal dipole singular at the origin. Further,

$$\phi_1(x, z) = \frac{\partial}{\partial x} G_0(x, z), \tag{C 2}$$

where G_0 denotes the real part of the wave source in the free surface discussed in Appendix B.2 of Linton & McIver (2001), and, in particular,

$$G_0(x, -h) = \frac{\pi \cosh kh \sin k|x|}{khN_0^2} + R(x), \tag{C 3}$$

where

$$N_0^2 = \frac{1}{2}\left(1 + \frac{\sinh 2kh}{2kh}\right)$$

and $R(x) \to 0$ as $|x| \to \infty$. Thus

$$\begin{aligned}\int_{-\infty}^{\infty} \phi_0(x, -h)\,\mathrm{d}x &= \lim_{X\to\infty} \int_{-X}^{X} \left(\frac{\partial}{\partial x} G_0(x - \xi, -h) - \frac{\partial}{\partial x} G_0(x + \xi, -h)\right) \mathrm{d}x \\ &= \lim_{X\to\infty} \Big[G_0(x - \xi, -h) - G_0(x + \xi, -h)\Big]_{-X}^{X} \\ &= \frac{\pi \cosh kh}{khN_0^2} \lim_{X\to\infty} \Big[\sin k|X - \xi| - \sin k|X + \xi|\Big]. \end{aligned} \tag{C 4}$$

For sufficiently large $|X|$

$$\begin{aligned}\sin k|X - \xi| - \sin k|X + \xi| &= \operatorname{sgn} X\, [\sin k(X - \xi) - \sin k(X + \xi)] \\ &= -2 \operatorname{sgn} X \cos kX \sin k\xi = 0 \end{aligned} \tag{C 5}$$

as $k\xi = n\pi$, $n \in \mathbb{Z}^+$, and hence (3.10) is satisfied.

Appendix D. Asymptotic analysis of the three-dimensional potential

In this appendix it is shown that the ring potential given in (4.7) and (4.8) has no far-field dipole coefficient. Any vertically axisymmetric potential ϕ that is wave free at infinity has, on $r = 0$, $\phi = \mathscr{D}/z^2 + O(z^{-2})$ as $z \to -\infty$ and so the absence of a dipole in the far field may be confirmed, without loss of generality, by taking $r = 0$ in the trapped-mode potential ϕ_0 and showing that $\phi_0 = o(z^{-2})$ as $z \to -\infty$.

When evaluated on $r = 0$, the ring-dipole potential contains the integral

$$I = \int_0^{\infty} (t \cos tz + K \sin tz) K_1(ct) \frac{t^2}{t^2 + K^2}\,\mathrm{d}t \tag{D 1}$$

(and a smaller exponentially decaying term which may be neglected in this analysis) and the aim here is to show that $I = o(z^{-2})$ as $z \to -\infty$.

We can write

$$I = I_1 + K I_2, \tag{D 2}$$

where

$$I_1 = \int_0^{\infty} \cos(zt) f_c(t)\,\mathrm{d}t, \quad I_2 = \int_0^{\infty} \sin(zt) f_s(t)\,\mathrm{d}t \tag{D 3}$$

and

$$f_c(t) = \frac{t^3}{t^2+K^2}K_1(ct), \quad f_s(t) = \frac{t^2}{t^2+K^2}K_1(ct). \tag{D 4}$$

Integration by parts twice gives

$$I_1 = \left[\frac{\sin zt}{z} f_c(t)\right]_0^\infty - \int_0^\infty \frac{\sin zt}{z} f_c'(t)\,\mathrm{d}t$$
$$= -\frac{1}{z}\left(-\left[\frac{\cos zt}{z} f_c'(t)\right]_0^\infty + \int_0^\infty \frac{\cos zt}{z} f_c''(t)\,\mathrm{d}t\right), \tag{D 5}$$

where, from the asymptotic forms of the modified Bessel function of the second kind, it is straightforward to verify that

$$\lim_{t\to 0} f_c'(t) = \lim_{t\to\infty} f_c'(t) = 0, \tag{D 6}$$

$$f_c''(t) \to \frac{1}{cK^2} \quad \text{as} \quad t \to 0, \tag{D 7}$$

and that $f''(t)$ is exponentially small as $t \to \infty$. Thus, $f_c''(t)$ is absolutely integrable and by the Riemann–Lebesgue lemma

$$\int_0^\infty \cos zt\ f_c''(t)\,\mathrm{d}t = o(1) \quad \text{as} \quad z \to \infty \tag{D 8}$$

so that $I_1 = o(z^{-2})$ as $z \to -\infty$.

Formal integration by parts twice yields

$$I_2 = \left[-\frac{\cos zt}{z} f_s(t)\right]_0^\infty + \int_0^\infty \frac{\cos zt}{z} f_s'(t)\,\mathrm{d}t, \tag{D 9}$$

but $f_s'(0) \neq 0$ and so an alternative approach must be taken. In this alternative, we can write

$$I_2 = \frac{1}{c}\int_0^\infty \frac{t\sin zt}{t^2+K^2}\,\mathrm{d}t + \int_0^\infty \sin zt\ f_{s2}(t)\,\mathrm{d}t \tag{D 10}$$

where the first integral is denoted I_{21}, the second I_{22} and the integrand in I_{22} is

$$f_{s2}(t) = \frac{t^2(K_1(ct) - 1/ct)}{t^2+K^2}. \tag{D 11}$$

A straightforward contour integration yields $I_{21} = -\pi\,\mathrm{e}^{Kz}/2c$ for $z < 0$ so that, in particular, $I_{21} = o(z^{-2})$ as $z \to -\infty$. The integral I_{22} is estimated through integration by parts and, in a similar way to I_1, it is found that

$$I_{22} = -\frac{1}{z^2}\int_0^\infty \sin zt\ f_{s2}''(t)\,\mathrm{d}t. \tag{D 12}$$

The term $f_{s2}''(t)$ is absolutely integrable and so by the Riemann–Lebesgue lemma $I_{22} = o(z^{-2})$ as $z \to -\infty$. Combining all of the above gives $I = o(z^{-2})$ as $z \to \infty$ and hence there is no dipole in the far-field expansion.

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